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On theories with light-like noncommutativity

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Abstract: We show that field theories with light-like noncommutativity, that is with $\theta^{0i} = -\theta^{1i}$, are unitary quantum theories, and that they can be obtained as decoupled field theory limits of string theory with D-branes in a background NS-NS B-field. For general noncommutativity parameters, we show that non-commutative field theories which are unitary can be obtained as decoupled field theory limits of string theory, while those that are not unitary cannot be obtained from string theory because massive open strings do not decouple. We study the different theories with light-like noncommutativity which arise from type-II D-branes. The decoupling limit of the D4-brane seems to lead to a non-commutative field theory deformation of the $(2,0)$ SCFT of M5-branes, while the D5-brane case leads to a non-commutative variation of "little string theories". We discuss the DLCQ description of these theories.

Keywords: Brane Dynamics in Gauge Theories, M(atrix) Theories, Conformal Field Models in String Theory.

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1. Introduction

Theories on non-commutative spaces, in which the coordinates satisfy $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, have been a very active topic of research in the last few years. They appear in decoupling limits of D-branes in string theory in backgrounds with non-zero NS-NS B-fields [1, 2, 3]. The initial research focused on theories with only space-like noncommutativity, that is with $\theta^{0i} = 0$. Gauge theories with space-like noncommutativity arise from a decoupling limit of string theory involving D-branes with non-zero space-like B-fields [3], in which all string modes decouple and one is left with a field theory (coming from the massless open strings ending on the D-branes). Field theories on such spaces are unitary.

Recently, it was realized that theories with time-like noncommutativity, that is $\theta^{0i} \neq 0$, may also exist. However, field theories on such spaces exhibit acausal behaviour [4, 5] and the quantum theories are not unitary [6]. In [7, 8, 9] it was found that a decoupled field theory limit for D-branes with a time-like B-field does not exist. However, references [7, 8] found a limit in which the closed strings decouple but the massive open strings do not, so this limit describes a non-commutative open string theory (NCOS) rather than a field theory. These open string theories were further analyzed in [10, 11]. Several related aspects were recently considered in [12]–[17].

In this paper we wish to analyze a third type of noncommutativity, in which the noncommutativity parameter $\theta^{\mu\nu}$ is light-like, for example with $\theta^{0i} = -\theta^{1i}$ (in light-cone coordinates this corresponds to $\theta^{i-} \neq 0$). We will argue that despite the nonlocality in the time coordinate due to $\theta^{0i} \neq 0$, field theories with light-like noncommutativity are quantum mechanically unitary and exhibit interesting properties.

In section 2 we determine which string backgrounds with a constant B-field admit a decoupled non-commutative field theory limit, and verify for a light-like B (say, for $B_{0i} = B_{1i} \neq 0$) that such a field theory limit exists. In section 3 we analyze perturbative unitarity of non-commutative field theories with arbitrary noncommutativity matrix $\theta^{\mu\nu}$. We show that non-commutative field theories which can be obtained as decoupled field theory limits of string theory are perturbatively unitary quantum theories. On the other hand, those non-commutative field theories that are not unitary cannot be obtained from string theory because massive open string modes do not decouple. Such theories can be made unitary by adding massive open string degrees of freedom, decoupled from the closed strings, and lead to NCOS theories. The relation between unitarity in field theory and decoupling in string theory is physically very appealing.

In section 4 we analyze the decoupling limits of D-branes in type-II string theory which lead to theories with light-like noncommutativity, decoupled from closed strings and from massive open strings. In the case of light-like noncommutativity, the open string coupling constant is identical to the closed string coupling constant. Therefore, the analysis of the decoupling limits is completely analogous to the analysis of decoupling limits of D-branes without a B-field. For D2-branes and D3-branes, we find decoupling limits giving $(2+1)$-dimensional and $(3+1)$-dimensional super-Yang Mills (SYM) theories (with light-like noncommutativity). The light-like non-commutative $(3+1)$-dimensional SYM theory exhibits a conventional field theoretic S-duality, such that the strong coupling limit of the non-commutative field theory on the D3-brane is also a non-commutative *field* theory with light-like noncommutativity.[1] For D4-branes we find that the decoupling limit seems to lead to a $(5+1)$-dimensional field theory (compactified on a circle), which is a non-commutative version of the $(2,0)$ six-dimensional SCFT.[2] For D5-branes we find in the decoupling limit a non-commutative version of "little string theories", which reduces to $(5+1)$-dimensional non-commutative SYM at low energies. Similar theories arise also from NS5-branes with non-zero light-like RR backgrounds. For the various six-dimensional theories we also describe the discrete light-cone quantization (DLCQ) of the light-like non-commutative theories, which is a simple variation of the DLCQ for the same theories on a commutative space.

[1] For a similar two-dimensional phenomenon see [18].

[2] Note that no such decoupled field theory exists for space-like fields, since the self-duality of the 3-form on the 5-brane forces a time-like noncommutativity to accompany any space-like noncommutativity.

2. Open strings and noncommutativity

Consider open strings on a single D-brane (the generalization to several overlapping D-branes is straightforward) in a constant background electromagnetic field (or, equivalently, in a constant background NS-NS two-form field) $B_{\mu\nu}$. The conformal field theory of this background was solved in [19, 20]. The dynamics of the open string is determined in terms of the sigma model metric (closed string metric) $g_{\mu\nu}$, the background two-form field $B_{\mu\nu}$ and the closed string coupling constant g_s. The signature of space-time will be taken to be $(-,+,\ldots,+)$. The propagator of open string worldsheet coordinates between boundary points τ and τ' on the real axis of the upper half-plane is[3]

$$\langle X^\mu(\tau)\, X^\nu(\tau')\rangle = -\alpha' G^{\mu\nu}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\,\mathrm{sign}(\tau-\tau')\,, \tag{2.1}$$

where

$$\begin{aligned} G^{\mu\nu} &= \left(\frac{1}{g+2\pi\alpha' B}\right)^{\mu\nu}_S = \left(\frac{1}{g+2\pi\alpha' B}g\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}, \\ \theta^{\mu\nu} &= 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)^{\mu\nu}_A = -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}, \end{aligned} \tag{2.2}$$

and the effective open string coupling is given by

$$G_o = g_s\sqrt{\frac{\det(g+2\pi\alpha' B)}{\det(g)}}\,. \tag{2.3}$$

The classical effective action on the D-brane is obtained from the S-matrix of open string states on the disc worldsheet. The presence of the term proportional to $\theta^{\mu\nu}$ in the propagator replaces the conventional product of fields in the effective action with the $\star$-product of fields.

We are interested in finding which electromagnetic backgrounds B admit a decoupled field theory limit such that the low energy effective description is given by a non-commutative field theory of the massless open string modes.[4] Moreover, we want to determine which non-commutative field theories are unitary quantum theories (see section 3). We will see that those four-dimensional non-commutative field theories that are perturbatively unitary are precisely those that can be obtained as a decoupled field theory limit of string theory. Moreover, the non-commutative field theories that are not unitary correspond to string backgrounds in which the non-commutative massless open strings do not decouple from the massive ones.

[3] Analogous expressions can be written for the worldsheet superpartners ψ^μ.

[4] Clearly, there is always a low energy limit whose description is given by conventional (commutative) field theory. Here we are interested in a non-commutative field theory description.

Our analysis will be based on looking at the Dirac-Born-Infeld action describing constant electromagnetic background fields and seeing when it describes a sensible theory on its own. We start by discussing the case of a D3-brane, for which we give a Lorentz-invariant description of the admissible backgrounds. Given a background $B_{\mu\nu}$ field, with particular values for the electromagnetic Lorentz invariants[5]

$$I_1 = \frac{1}{2} B_{\mu\nu} B^{\mu\nu} = \mathbf{B}^2 - \mathbf{E}^2 \,, \qquad I_2 = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma} = \mathbf{E} \cdot \mathbf{B} \,, \tag{2.4}$$

one can perform a Lorentz transformation to go to a standard frame where it is simple to study the existence of a decoupled field theory limit. In the standard frame, $\mathbf{E}$ can be chosen to be parallel, anti-parallel or orthogonal to $\mathbf{B}$. There are 9 separate possibilities depending on I_1 and I_2. The standard frames are:

1. $I_1 > 0 \; I_2 > 0 : \; \mathbf{E} \| \mathbf{B}, \, \mathbf{B}^2 > \mathbf{E}^2 \,;$
2. $I_1 > 0 \; I_2 < 0 : \; -\mathbf{E} \| \mathbf{B}, \, \mathbf{B}^2 > \mathbf{E}^2 \,;$
3. $I_1 < 0 \; I_2 > 0 : \; \mathbf{E} \| \mathbf{B}, \, \mathbf{B}^2 < \mathbf{E}^2 \,;$
4. $I_1 < 0 \; I_2 < 0 : \; -\mathbf{E} \| \mathbf{B}, \, \mathbf{B}^2 < \mathbf{E}^2 \,;$
5. $I_1 = 0 \; I_2 > 0 : \; \mathbf{E} \| \mathbf{B}, \, \mathbf{B}^2 = \mathbf{E}^2 \,;$
6. $I_1 = 0 \; I_2 < 0 : \; -\mathbf{E} \| \mathbf{B}, \, \mathbf{B}^2 = \mathbf{E}^2 \,;$
7. $I_1 > 0 \; I_2 = 0 : \; \mathbf{E} \perp \mathbf{B}, \, \mathbf{B}^2 > \mathbf{E}^2 \,;$
8. $I_1 < 0 \; I_2 = 0 : \; \mathbf{E} \perp \mathbf{B}, \, \mathbf{B}^2 < \mathbf{E}^2 \,;$
9. $I_1 = 0 \; I_2 = 0 : \; \mathbf{E} \perp \mathbf{B}, \, \mathbf{B}^2 = \mathbf{E}^2 \,.$

It is known [3] that a space-like non-commutative field theory can be obtained as a decoupled limit of background (7), since one can always go to a frame in which only the $\mathbf{B}$ field is non zero. Moreover, background (8) can be boosted to a frame in which only the $\mathbf{E}$ field is non zero, and [7, 8, 9] showed that no decoupled field theory limit exists for this background. It is easy to see that whenever $\mathbf{E}$ is either parallel or antiparallel to $\mathbf{B}$ (backgrounds (1)–(6)) there is no decoupled non-commmutative field theory limit. The physical origin for the nonexistence of a decoupled field theory limit is that in order to decouple the theory one must take both $\mathbf{B}$ and $\mathbf{E}$ large [3], but whenever $I_2 \neq 0$ there is an upper critical value of the electric field E_c beyond which the theory becomes unstable and, therefore, no sensible decoupled field theory exists. In such a background the $\mathbf{E}$ field reduces the tension of a string when the string is stretched in the direction of $\mathbf{E}$, and it becomes tensionless precisely at E_c.

[5] We take $B_{0i} = \mathbf{E}_i$ and $B_{ij} = \epsilon_{ijk} \mathbf{B}_k$.

Having a parallel (anti-parallel) **B** field does not change this phenomenon. More explicitly, consider the Dirac-Born-Infeld lagrangian density for a single D-brane in a background metric $g_{\mu\nu} = \mathrm{diag}(-g, g, g, g)$ and with arbitrary background **B** and **E** fields,

$$\begin{aligned} \mathcal{L}_{\mathrm{DBI}} &= -T_3\sqrt{-\det(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})} \\ &= -T_3\sqrt{g^4 + (2\pi\alpha')^2 g^2(\mathbf{B}^2 - \mathbf{E}^2) - (2\pi\alpha')^4(\mathbf{E}\cdot\mathbf{B})^2}\,. \end{aligned} \tag{2.5}$$

Clearly, whenever $I_2 \neq 0$ the theory becomes unstable for $|\mathbf{E}| > E_c \equiv g/(2\pi\alpha')$ and, therefore, there is no decoupled non-commutative field theory limit.

The only case left to consider is when $I_1 = I_2 = 0$ (note that one cannot always transform this case to $\mathbf{E} = \mathbf{B} = 0$, except by an infinite Lorentz boost). This is the light-like non-commutative case, where $\mathbf{E}^2 = \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B} = 0$. Clearly, there is no obstruction to taking the decoupled field theory limit since there is no instability for any value of the **E** field. In this case, the presence of the **B** field perpendicular to **E** forbids the **E** field from reducing the energy of the string so that it becomes tensionless. Summarizing, the Lorentz invariant criterion for backgrounds from which one can find a four-dimensional decoupled field theory limit is $I_1 \geq 0$ and $I_2 = 0$. The remaining backgrounds can be made unitary by adding massive open string degrees of freedom, decoupled from closed strings, and can lead to NCOS theories in an appropriate limit. This criterion will be recovered in the following section from a field theoretic analysis of unitarity.

Similarly, it is easy to show for any Dp-brane with $p \geq 2$ that a light-like non-commutative field theory can also be obtained from string theory in a background NS-NS B-field $B_{0i} = B_{1i}$. For D2-branes the only Lorentz-invariant that can be constructed from the background field is $I_1 = \frac{1}{2}B_{\mu\nu}B^{\mu\nu}$. The possible cases are $I_1 > 0$ leading to the usual non-commutative Yang-Mills theory, $I_1 < 0$ leading to the non-commutative open string theory, and $I_1 = 0$ which is the light-like case that we will discuss here.

3. Unitarity constraints

In [6] unitarity of space-like non-commutative field theories and time-like non-commutative theories was studied at the one loop level, and it was found that space-like non-commutative theories are unitary while time-like non-commutative theories are not unitary. One can easily perform a general analysis of which types of noncommutativity lead to unitary theories and which do not.

Unitarity requires [6] that the inner product $p \circ p$ is never negative, where p is some external momentum and

$$p \circ p \equiv -p_\mu \theta^{\mu\rho} G_{\rho\sigma} \theta^{\sigma\nu} p_\nu \equiv p_\mu g_\theta^{\mu\nu} p_\nu \geq 0\,, \tag{3.1}$$

where $\theta^{\mu\nu}$ is the noncommutativity matrix and $G_{\rho\sigma}$ is the background metric of the field theory. The reason behind this requirement is that in order to define loop integrals in these theories, one must analytically continue the momentum and $\theta^{\mu\nu}$ to euclidean space such that the euclidean expression for $p \circ p$ is positive,[6] so that Feynman graphs are well defined. In order to check unitarity of the theory one must analytically continue answers to Minkowski space. Therefore, if in Minkowski space $p \circ p < 0$, Green's functions acquire branch cuts as a function of momentum. It is the presence of these extra branch cuts[7] that causes nonunitary answers, since they lead to extra imaginary pieces for S-matrix elements that violate the optical theorem.

We will analyze in detail the four-dimensional case and comment below on the other cases. A necessary condition for unitarity is that the eigenvalues of $g_\theta^{\mu\nu}$ are nonnegative. This ensures that $p \circ p \geq 0$ and that no unphysical branch cuts in Green's functions appear. Therefore, we demand that

$$\det(g_\theta^{\mu\nu}) = \det(-\theta^{\mu\rho} G_{\rho\sigma} \theta^{\sigma\nu}) \geq 0\,. \tag{3.2}$$

It is useful to rewrite the background metric of the field theory as

$$G_{\mu\nu} = \big((g - 2\pi\alpha' B) g^{-1} (g + 2\pi\alpha' B)\big)_{\mu\nu}\,. \tag{3.3}$$

Using (2.2) it follows that

$$\det(g_\theta^{\mu\nu}) = (2\pi\alpha')^4 \det\left(-\frac{1}{g + 2\pi\alpha' B} B g^{-1} B \frac{1}{g - 2\pi\alpha' B}\right). \tag{3.4}$$

Using the fact that $\det(g + 2\pi\alpha' B) = \det(g - 2\pi\alpha' B)$ one gets

$$\det(g_\theta^{\mu\nu}) = (2\pi\alpha')^4 \frac{1}{\det^2(g + 2\pi\alpha' B)\det(-g)} \overset{2}{\det}(B)\,. \tag{3.5}$$

Now, since $\det(-g) < 0$ and $\det^2(g + 2\pi\alpha' B) \geq 0$, and

$$\overset{2}{\det}(B) = (\mathbf{E} \cdot \mathbf{B})^4 = I_2^4\,, \tag{3.6}$$

a necessary condition for unitarity is that

$$I_2 = \mathbf{E} \cdot \mathbf{B} = 0\,. \tag{3.7}$$

[6] We will avoid values of the external momenta for which $p \circ p = 0$, which lead to peculiar infrared divergences.

[7] Green's functions in these theories also have the conventional physical branch cuts associated with threshold production of multiparticle states.

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Therefore, there are three cases to be considered that can lead to a unitary quantum field theory:[8]

(7) In this case one can transform to a frame in which only the **B**-field is non-zero, for example $B_{12} \neq 0$. This leads to space-like noncommutativity with $\theta^{12} = \theta$. Then, we have $p \circ p = \theta^2(p_1^2 + p_2^2) \geq 0$ and the theory is unitary.

(8) In this case one can go to a frame in which only the **E**-field is non-zero, for example $B_{01} \neq 0$. This leads to time-like noncommutativity with $\theta^{01} = \theta$. Then, $p \circ p = \theta^2(p_0^2 - p_1^2)$ can be negative and the theory is not unitary.

(9) In this case one can go to a frame with $B_{02} = B_{12}$. This leads to light-like noncommutativity with $\theta^{02} = -\theta^{12} = \theta$. Then, $p \circ p = \theta^2(p_0 - p_1)^2 \geq 0$ and the theory is unitary.

Therefore, there is precise agreement between the backgrounds which have a decoupled non-commutative field theory limit and the field theories which have a perturbatively unitary S-matrix. It is easy to generalize this also to other dimensions: the behavior of $p \circ p$ in the presence of space-like, time-like and light-like noncommutativity is always as in the cases (7), (8) and (9) discussed above (respectively). In the rest of the paper we will concentrate on theories with light-like noncommutativity.

4. Decoupling limit with light-like noncommutativity

In the previous two sections we showed that there could exist a decoupled light-like non-commutative field theory limit of string theory, and that the resulting field theory is quantum mechanically unitary. In this section we will study this decoupled field theory limit in detail for all D-branes of type-II string theory. It is convenient to analyze such decoupled field theories in light-cone coordinates, $x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$.

By a Lorentz transformation we can always choose the light-like noncommutativity parameter to be $\theta^{2-} \equiv \Theta \neq 0$, with all other noncommutativity parameters vanishing. In the usual coordinates such noncommutativity appears whenever $\theta^{20} = -\theta^{21} = \Theta/\sqrt{2}$. Such a configuration involves noncommutativity in the time direction ($\theta^{20} \neq 0$), which results in a theory non-local in time. Naively, one would not expect such a theory to be unitary, nor would one expect that it can be obtained from a decoupled limit of string theory. However, we can always choose to perform a light-cone quantization in which x^+ is the time coordinate. The field theory is local in the x^+ time coordinate since $\theta^{i+} = 0$. Therefore, one would expect the light-cone hamiltonian $H \equiv P_+$ to be hermitean, and the field theory to be well defined. In this section we describe how to get a field theory with this type of noncommutativity as a limit of string theory.

[8] We will take the open string metric to be $G^{\mu\nu} = \eta^{\mu\nu}$ in the equations below, a different metric with the same signature will lead to the same results.

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We start with k Dp-branes with general p, but we will focus only on the first three coordinates[9] since the others will always have a flat metric and no background fields. We take the closed string metric to be the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, and turn on a non-zero B_{2+}. Using (2.2) [3], we find the open string metric $G^{+-} = -G^{22} = -1$, $G^{--} = -(2\pi\alpha' B_{2+})^2$, and the noncommutativity parameter $\theta^{2-} = (2\pi\alpha')^2 B_{2+}$. We wish to discuss a decoupling limit in which we take $\alpha' \to 0$ to decouple the closed strings and the massive open strings. In order to obtain a finite noncommutativity parameter $\theta^{2-} \equiv \Theta$ in the gauge theory we need to take a very large $B_{2+} = \Theta/(2\pi\alpha')^2$. Equivalently, one can turn on a constant flux in the overall U(1) factor in the D-brane gauge group, $F_{2+} = \Theta/(2\pi\alpha')^2$ (times the identity matrix). This requires taking a very large electric field E_2. As discussed in section 2, when the background flux is light-like, a large electric field does not lead to an instability.

At first sight, we end up in this limit with a strange open string metric with an infinite G^{--} component. However, this does not actually have any physical effect, and we can easily fix this[10] by a change of coordinates

$$y^+ \equiv x^+ ; \qquad y^- \equiv x^- + \frac{1}{2} G^{--} x^+ ; \qquad y^i \equiv x^i \ (i = 2, \ldots, p) . \tag{4.1}$$

In the new coordinates the open string metric is $G^{\mu\nu} = \eta^{\mu\nu}$ and we have a finite noncommutativity parameter $\theta^{2-} = \Theta$, so we obtain precisely the field theories discussed above. Equivalently, we could have started with a closed string metric with g_{++} which goes to infinity such that the open string metric is diagonal; this situation is related to the situation we describe here by a shift similar to (4.1).

It is important to note that the theories with light-like noncommutativity which we discuss here do not have a typical noncommutativity scale in them, since there is no Lorentz-invariant scalar one can make out of θ^{2-}. Longitudinal Lorentz boosts can rescale θ^{2-} to any (non-zero) value we wish it to be. The scaling of B_{2+} which we describe above is the one which gives $\theta^{2-} = \Theta = \textit{constant}$ in the decoupling limit, but any scaling of these parameters (which gives a non-zero and finite θ^{2-}) is related by a boost to the scaling we describe above. Correlation functions in these theories depend on the longitudinal boost invariant combination $\theta^{2-} P_-$.

Using (2.3) we find that the open string coupling constant in this case is the same as it was without the B-field, namely $G_o = g_s$, so that the Yang-Mills (YM) coupling constant is given by the usual formula $g_{\rm YM}^2 = (2\pi)^{p-2} g_s (\alpha')^{(p-3)/2}$. The discussion of the possible decoupling limits is thus exactly the same as without the B-field and not the same as in the case of a space-like B-field. One scales $\alpha' \to 0$ to decouple the field theory from the bulk and scales g_s such that one is left with a non-trivial field theory on the brane ($g_{\rm YM}$ is kept fixed). We will now analyze the decoupled theories that we get in different dimensions:

[9] In order to have a light-like non-commutative field theory $p \geq 2$.
[10] This was suggested to us by N. Seiberg.

4.1 D3-branes

For k D3-branes we can take $\alpha' \to 0$ keeping g_s fixed, and we get a U(k) NCYM theory with finite noncommutativity, which is decoupled from the closed strings and from the massive open strings by the same arguments used in the absence of the B-field.

It is interesting to note that in the $(3+1)$-dimensional case the U(k) gauge theories that we find go to themselves under S-duality, unlike the theories with space-like noncommutativity which are S-dual to non-commutative open string theories (NCOS) [8]. In the light-like case the $(3+1)$-dimensional decoupled theories inherit the S-duality transformation from type-IIB string theory. This transformation inverts the (complexified) gauge coupling and changes the background flux; to leading order in the background flux it exchanges $F_{\mu\nu}$ with $(*F)_{\mu\nu}$ [21], where $*$ denotes the Hodge operation, and for light-like fields this is actually the exact transformation. This leads to a field theory with a light-like noncommutativity parameter θ^{3-}. Generally, S-duality changes the light-like noncommutativity parameter by $\theta^{i-} \to \epsilon^{ij}\theta^{j-}$, where the epsilon symbol involves the directions transverse to the light-cone coordinates.

4.2 D2-branes

For k D2-branes, if we want to keep the YM coupling constant fixed as we take $\alpha' \to 0$ we must also scale $g_s \propto (\alpha')^{1/2} \to 0$ at the same time, but this obviously does not affect the decoupling arguments. In this limit we find precisely the $(2+1)$-dimensional U(k) light-like non-commutative supersymmetric gauge theory.

4.3 D4-branes

Things become more interesting if we discuss the decoupling limit for k D4-branes. In this case, if we wish to take $\alpha' \to 0$ and keep the YM coupling constant fixed, we must scale g_s to infinity as $(\alpha')^{-1/2}$. Thus, it is more appropriate to think of the theory as M-theory compactified on a circle. The Planck scale in M-theory scales as $M_p^3 = M_s^3/g_s \propto (\alpha')^{-1}$ so it goes to infinity, while the radius of the M-theory circle remains finite (as in the absence of the B-field), $R_{11} = g_s(\alpha')^{1/2} \simeq g_{\mathrm{YM}}^2$. The decoupled theory on the D4-branes should thus be viewed as a decoupled theory on k M5-branes compactified on a finite circle. This is not surprising since the $(4+1)$-dimensional gauge theory on its own is non renormalizable even before we add the noncommutativity.

When we go to M-theory it is natural to keep the metric on the brane (which is the same as the metric in the bulk up to an infinite g_{++} which we discussed above) in the form $G^{\mu\nu} = \eta^{\mu\nu}$. In these coordinates the x^{11} direction has periodicity $2\pi R_{11}$. Translating the relation $B_{2+} \simeq \Theta/(2\pi\alpha')^2$ to M-theory variables, we find that the

3-form of M-theory scales as

$$C_{2(11)+} \simeq -\Theta R_{11} M_p^6 \tag{4.2}$$

in the limit we are taking, with R_{11} constant and M_p going to infinity.

We claim that this limit, for k M5-branes oriented in the $(0,1,2,3,4,11)$ directions, defines a decoupled "non-commutative" variation of the $(2,0)$ theory living on the M5-branes. In M-theory, one can gauge away any constant components of the background C-field that are transverse to the M5-branes, as well as the anti-self-dual components of C along the M5-branes. So, we can take the background C-field to be self-dual, with non-vanishing $C_{34+} = C_{2(11)+}$. Equivalently, instead of the C-field we can take the self-dual 3-form worldvolume field $H_{34+} = H_{2(11)+}$ on the M5-branes to scale in the same way that we scaled the C-field in the decoupling limit. Here, we used the fact that for light-like fields the non-linear self-duality condition on the 3-form field H actually becomes linear [3].

It is not clear how to characterize the "noncommutativity" (or whatever generalizes this notion) in the six-dimensional theory. It seems reasonable to expect that this theory has a 3-form "generalized noncommutativity parameter", which would be (for example) the coefficient of the leading (dimension 9) irrelevant operator appearing in the low-energy expansion of the theory. If we call this parameter $\psi^{\mu\nu\rho}$, dimensional analysis and Lorentz covariance determine that in the light-like "non-commutative" case described above it will be given by $\psi^{2(11)-} = -C_{2(11)+}/M_p^6 \simeq \Theta R_{11}$. This means if we take the $R_{11} \to \infty$ (or $g_{\mathrm{YM}} \to \infty$) limit in the theory described above, we do not get a theory with finite "noncommutativity". Rather, such a theory would arise from taking $C_{2(11)+} \simeq -\psi M_p^6$ with ψ kept constant as $M_p \to \infty$. However, since we do not understand the notion of "generalized noncommutativity" we cannot rigorously justify these claims. In [12, 13] it was suggested that six-dimensional "non-commutative" theories can be characterized by an open membrane metric which could be analogous to the open string metric described above; in our case this "open membrane" metric turns out to be $\eta^{\mu\nu}$, just like the open string metric on the D4-brane. The fact that the "open membrane metric" remains finite as we take M_p to infinity is consistent with our claim the the six-dimensional theory is a field theory, with no additional open strings or membranes.

A theory which seems to describe the DLCQ of the six-dimensional theory described above was discussed in [22, section 4]. The decoupled theory of k M5-branes with N units of light-like momentum ($P_- = N/R$) was described in terms of the $g_{\mathrm{YM}} \to \infty$ limit of the Higgs branch of the $\mathcal{N} = 8$ U(N) $(0+1)$-dimensional SYM theory with k hypermultiplets in the fundamental representation [23], and the Fayet-Iliopoulos (FI) parameters of this theory were identified (in a particular normalization) with C_{ij+}/RM_p^6 (where R is the radius of the compact light-like direction). Note that in the DLCQ, where the x^- direction is compact, we can no longer perform arbitrary longitudinal Lorentz boosts since these also rescale the radius R; the combination C_{ij+}/R appearing in the DLCQ is boost-invariant and can thus be used to characterize the "noncommutativity" of the theory. The fact that the DLCQ depends on C_{ij+}/M_p^6 is consistent with our conjecture for the "generalized noncommutativity parameter" described above. The infinite shift we found (4.1) between the closed string and open string coordinates can be identified with the infinite shift found in the DLCQ between the vacuum energies of the Higgs and Coulomb branches (in the decoupling limit).

The relation between the six-dimensional theory we described here and the "open membrane" theories discussed in [10, 12, 13] is not clear. Those theories involve additional degrees of freedom in addition to the six-dimensional field theory, while such degrees of freedom do not seem to appear in our case.

4.4 D5-branes and NS5-branes

For k D5-branes, again we have to take g_s to infinity as we take α' to zero, in order to keep the Yang-Mills coupling fixed. The strong coupling limit of type-IIB string theory is described by the S-dual theory, in which the string coupling goes to zero. Thus, it is best to describe the limit we are discussing in the S-dual theory. In this theory we find that we have k NS 5-branes, the string coupling goes to zero, and the string tension (which is the inverse gauge coupling on the NS5-branes) remains constant. This is the same limit used to define "little string theories" (LSTs) [24, 25, 26], so the theory we get in this limit is a non-commutative version of the LSTs. The S-duality turns the NS-NS B-field into a RR B-field. Therefore, we are discussing NS 5-branes with a constant RR B_{2+} field which goes to infinity. Equivalently (as in the previous cases) we can just take the gauge field strength F_{2+} on the 5-branes to go to infinity. At low energies (compared to the string tension) this limit gives a (non-renormalizable) light-like non-commutative $(5+1)$-dimensional gauge theory, while at energies of the order of the string scale we have the full non-commutative LST.

As in the previous discussion, the DLCQ of this NCLST is given by a simple deformation of the DLCQ of the LST with $(1,1)$ supersymmetry [27, 28]. This DLCQ description (which is reviewed in [29]), for the theory of k 5-branes with N units of light-like momentum, is given by the low-energy SCFT of the Coulomb branch of the $(1+1)$-dimensional $\mathrm{U}(N)^k$ gauge theory with bifundamental hypermultiplets for consecutive $\mathrm{U}(N)$ groups (arranged in a circle). The non-commutative deformation is realized in the DLCQ by adding an equal mass to the k bifundamental hypermultipets. Note that this mass, like the light-like non-commutative parameter, is a vector of the $\mathrm{SO}(4)$ rotation group acting on the four transverse coordinates of the 5-branes.

A similar deformation exists also for the $(2,0)$-supersymmetric LST arising from NS 5-branes in type-IIA string theory. The "non-commutative" deformation now involves a constant RR 3-form field C_{ij+}, or equivalently a constant 3-form field in the 5-brane worldvolume. In the DLCQ this deformation corresponds (as discussed in [29]) to turning on a Fayet-Iliopoulos term in the corresponding (1+1)-dimensional gauge theory [23, 30]. At low energies (compared to the string scale) the $(2,0)$ "non-commutative" LST reduces to the $(2,0)$ "non-commutative" field theory arising from k M5-branes, which we described in the previous subsection.

For higher-dimensional D-branes there seems to be no decoupling limit from the bulk, just like in the case without the noncommutativity.

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